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Abstract. The integrals of motion for a cylindrically symmetric stationary vortex are obtained in a covariant description of a mixture of interacting superconductors, superfluids and normal fluids. The relevant integrated stress–energy coefficients for the vortex with respect to a vortex–free reference state are calculated in the approximation of a "stiff", i.e. least compressible, relativistic equation of state for the fluid mixture. As an illustration of the foregoing general results, we discuss their application to some of the well known examples of "real" superfluid and superconducting systems that are contained as special cases. These include Landau's two–fluid model, uncharged binary superfluid mixtures, rotating conventional superconductors and the superfluid neutron–proton–electron plasma in the outer core of neutron stars.

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I. INTRODUCTION

The subject of investigation in the present work is the structure and energy of a stationary and cylindrically symmetric quantized vortex in an interacting multi–fluid mixture, which may consist of charged and uncharged superfluids and of normal fluids. This analysis has initially been motivated by the superfluid mixture commonly found in neutron star models, namely in the outer core region, where superfluid neutrons, superconducting protons and normal electrons are generally thought to coexist. However, due to the generality of the present approach, it is equally well applicable to superfluid and superconducting systems found in more common laboratory contexts, some of which will be discussed briefly in the concluding section VIII.

The study of superfluid mixtures has a long history, beginning with the pioneering work of Khalatnikov [1], later followed by the analysis of Andreev and Bashkin [2], who incorporated allowance for a (nondissipative) interaction between the superfluids. This effect is called "entrainment" (sometimes also "drag") and plays a central role in the study of such fluid mixtures. The model has been further extended by Vardanian and Sedrakian [3] to include charged fluids, and later a Hamiltonian formulation in the Newtonian framework has been developed by Mendell and Lindblom [4]. The problem of vortices in such mixtures has been considered especially in the context of neutron stars, namely by Sedrakian and Shahabasian [5], Alpar, Langer, Sauls [6], Mendell [7] and others.

The covariant vortex solution in a single uncharged superfluid has been analyzed by Carter and Langlois [8], who have also considered the modifications due to the compressibility of the superfluid. The present work is on the one hand a generalization of this analysis to arbitrary fluid mixtures, including charged ones and their coupling to electromagnetic fields, but on the other hand is restricted (for technical reasons) to the case of a "stiff" equation of state. This "stiff" case is characterized by the speed(s) of sound being equal to the speed of light, and is, within the limits of causality, the closest analogue to the common Newtonian incompressible models. Compressibility effects will be subject of future work. Finally, we mention the previously found result [9] for a Newtonian vortex in a rotating superconductor, that the (hydrodynamic) vortex energy is strictly independent of the rotating "normal fluid" of positively charged ions, a result that will be found here to hold under much more general conditions.

In the present work we will consider only stationary situations, which has two major advantages. First, it restricts the normal fluids to be in a state of *rigid* motion, and moreover in the *same* state of rigid motion, because normal fluids always possess some nonvanishing amount of viscosity and mutual friction. This even allows to describe a solid component in the present framework as a "normal fluid", because in the rigid state of motion the anisotropic effects of viscosity and elasticity become irrelevant. So we can for example conveniently describe a conventional laboratory superconductor as a superconducting–normal fluid mixture, consisting of superconducting electrons and a "normal" lattice of ions, as will briefly be discussed in the concluding section. The second and even more powerful consequence of stationarity is that we can use a *conservative* model based on a Lagrangian formalism that has been developed in recent years [10,11] in a generally covariant language. The use of a generally covariant instead of simply Newtonian description has also been motivated initially by the perspective of application to neutron stars, where relativistic effects inevitably come into play, but this approach turns out to be generally more flexible and convenient for the hydrodynamic description of such systems, even if relativistic effects are not important.

The plan of this work is as follows. In Sec. II we introduce the relevant notions and equations of the covariant multi–fluid formalism on which the present analysis is based. In Sec. III we discuss the description of superfluids in this framework and the topology of the vortex–type configurations. Sec. IV introduces what we called the "mongrel" representation of superfluid–normal mixtures, that consists of choosing the *superfluid momenta* and the *normal currents* as the basic variables of the description, and which will be particularly convenient for the present problem. In Sec. V we specify the class of cylindrically symmetric and stationary vortex configurations and obtain the first integrals of motion for these solutions. Sec. VI is devoted to the specification and the properties of the reference state, needed to separate the quantities attributed to the vortex from the fluid background. Finally, the relevant vortex stress–energy coefficients are integrated in Sec. VII, using the most general hydrodynamic modelization for the vortex core, and we find that the "rotation energy cancellation lemma" of [9] still holds under the more general conditions of the present work. In the concluding section VIII, we briefly illustrate the application of the foregoing results to some of the well known examples of superfluid and superconducting systems.

II. COVARIANT DESCRIPTION OF PERFECT FLUID MIXTURES

The general class of (non–dissipative) mixtures of charged or neutral perfect fluids has been shown by Carter [10] to be describable by an elegant covariant action principle. In this section we will briefly introduce the part of the formalism and notations that will be relevant to the present work.

In the absence of electromagnetic effects, a mixture of perfect fluids can be described by a Lagrangian density Λ that depends only on the particle number currents n_X^α, where late Latin indices, X, Y etc., enumerate the different fluid constituents. Variation of Λ with respect to the currents,

$$\delta\Lambda = \mu^X_\alpha \, \delta n_X^\alpha \,, \quad \text{i.e.} \quad \mu^X_\alpha \equiv \frac{\partial \Lambda}{\partial n_X^\alpha} \,, \tag{1}$$

defines the *dynamical* momenta per particle μ^X_α as the conjugate variables of the currents n_X^α with respect to Λ. Here and in the following we use implicit summation (except otherwise stated) over identical spacetime as well as constituent indices. Legendre transformation with respect to the currents, i.e.

$$\mathcal{P} \equiv \Lambda - n_X^\alpha \, \mu^X_\alpha \,, \tag{2}$$

defines the "Hamiltonian density" $\mathcal{P}$ as a function of the dynamic momenta μ^X_α. This function only exists for nondegenerate systems, that is, if the functions $\mu^X_\alpha(n_Y^\beta)$ defined in (1) are invertible. The conjugate relations can then be written as

$$n_X^\alpha = -\frac{\partial \mathcal{P}}{\partial \mu^X_\alpha} \,. \tag{3}$$

Furthermore, the form of these relations is constrained by the requirement of covariance, namely $\mathcal{P}$ (as well as Λ) has to be a *scalar* density, and can therefore only depend on scalars, i.e. on $\mu^X_\alpha \mu^{Y\alpha}$. This restricts relation (3) to be of the form

$$n_X^\alpha = K_{XY} \, \mu^{Y\alpha} \,, \tag{4}$$

where the (necessarily symmetric) matrix K_{XY} is defined as

$$K_{XY} \equiv -2 \frac{\partial \mathcal{P}}{\partial (\mu^X_\alpha \mu^{Y\alpha})} \,, \tag{5}$$

The condition of a non–degenerate system is equivalent to $\det K \neq 0$, and so we can write the inverse relation

$$\mu^X_\alpha = K^{XY} \, n_{Y\alpha} \,, \quad \text{with} \quad K^{XY} \, K_{YZ} \equiv \delta^X_Z \,. \tag{6}$$

In the case of noninteracting fluids, the Hamiltonian $\mathcal{P}$ would not depend on crossed scalars $\mu^X_\alpha \mu^{Y\alpha}$ with $X \neq Y$, but only on diagonal terms $(\mu^X_\alpha \mu^{X\alpha})$. In this case the matrix K_{XY} would be diagonal, and each current would be aligned with the respective momentum, similar to the case of a single perfect fluid, but any interaction terms between different fluid constituents in the Hamiltonian will lead to nondiagonal components of K_{XY}, and therefore the currents will become linear combinations (in each point) of the respective momenta. This (nondissipative) effect is called "entrainment" and has first been considered for superfluid mixtures of ^{3}He and ^{4}He by Andreev and Bashkin [2].

Before we come to the equations of motion, we need to extend our description to include the electromagnetic field and its coupling to charged fluids. This is done via the standard "minimal coupling" prescription that consists of defining the *total* Lagrangian density $\mathcal{L}$ as

$$\mathcal{L} \equiv \Lambda + j^\alpha A_\alpha + \frac{1}{16\pi} F_{\alpha\beta} F^{\beta\alpha} \,, \tag{7}$$

where we are using units with $c = 1$. The electric current j^α is defined as

$$j^\alpha \equiv e^X n_X^\alpha \,, \tag{8}$$

with e^X being the charge per particle of the constituent X. The electromagnetic 2–form $F_{\alpha\beta}$ is defined as the exterior derivative of the gauge 1–form A_α, i.e.

$$F_{\alpha\beta} \equiv 2\nabla_{[\alpha} A_{\beta]} \,, \tag{9}$$

where square brackets indicate (averaged) index antisymmetrization. The symbol ∇_α denotes the usual covariant derivative, but we note that because of the antisymmetrization, exterior derivatives are *independent* of the affine connection, so we could as well replace ∇_α by the partial derivative ∂_α.

The conjugate variables of the currents n_X^α with respect to the *total* Lagrangian $\mathcal{L}$ are the *canonical* momenta π^X_α, defined as

$$\pi^X_\alpha \equiv \frac{\partial \mathcal{L}}{\partial n_X^\alpha} \,, \tag{10}$$

which can be seen from (1) and (7) to be directly related to the dynamical momenta μ^X_α, namely

$$\pi^X_\alpha = \mu^X_\alpha + e^X A_\alpha \,. \tag{11}$$

The equations of motion are to be derived from the total Lagrangian $\mathcal{L}$ via an appropriate variational principle. Imposing invariance of the action under free (infinitesimal) variations of the gauge field A_α leads to the Maxwell source equation,

$$\nabla_\beta F^{\alpha\beta} = 4\pi j^\alpha \,. \tag{12}$$

However, the equations of motion for the fluids can not be derived via free variations of the currents n_X^α, as this would simply lead to the trivial equations $\pi^X_\alpha = 0$. This is because free variations of the currents contain too many degrees of freedom, which results in overdetermined equations of motion, therefore the variations have to be *constrained*. It has been shown in [10] that variations δn_X^α with the correct number of degrees of freedom are generated by infinitesimal displacements of the worldlines of fluid particles. These worldline variations satisfy the physical constraint of conserving the number of particles, and they result in the correct equations of motion for the fluids. Without entering into the technical details of this procedure (see [10,11]), the resulting equation of motion for each fluid X is found as (no sum over X)

$$2\, n_X^\alpha \nabla_{[\alpha} \pi^X_{\beta]} + \pi^X_\beta \, \nabla_\alpha n_X^\alpha = 0 \,, \tag{13}$$

and by contracting this equation with n_X^β, we see that it implies that the currents are conserved, i.e. $\nabla_\alpha n_X^\alpha = 0$, so the equations of motion reduce to the simple form of a vorticity conserving flow, namely (no sum over X)

$$n_X^\alpha \, w^X_{\alpha\beta} = 0 \,, \tag{14}$$

where the (canonical) vorticity 2–form $w^X_{\alpha\beta}$ is defined as the exterior derivative of the canonical momentum π^X_α, i.e.

$$w^X_{\alpha\beta} \equiv 2\nabla_{[\alpha} \pi^X_{\beta]} \,. \tag{15}$$

The very compact form (14) of the equation of motion can be seen to "reduce" in the nonrelativistic limit to the (much less compact) Euler equation of a charged fluid in electromagnetic fields, and possibly subject to further potential forces. This is an example that shows the advantage and convenience of the covariant formalism, especially for more complex applications like interacting mixtures of possibly charged fluids in electromagnetic fields, as considered in the present analysis.

And finally, the stress–energy tensor $T^{\alpha\beta}$ is found [11] in the form

$$T^{\alpha}{}_{\beta} = n_X^{\alpha}\mu^X_{\beta} + \mathcal{P}g^{\alpha}{}_{\beta} + \frac{1}{4\pi}\left(F^{\alpha\lambda}F_{\beta\lambda} - \frac{1}{4}F^{\rho\lambda}F_{\rho\lambda}\,g^{\alpha}{}_{\beta}\right), \tag{16}$$

which (in the absence of external forces) satisfies the equation of (pseudo) conservation, $\nabla_\alpha T^{\alpha\beta} = 0$. From the form of the stress–energy tensor (16) we see that $\mathcal{P}$ plays the role of a generalized pressure, which reduces to the ordinary pressure in the case of a single fluid.

III. PROPERTIES OF SUPERFLUIDS AND TOPOLOGY OF VORTEX SOLUTIONS

We want to allow for some of the fluids to be superfluid or superconducting, and we will denote these constituents by capital Greek indices Υ, Ψ etc. For "normal" fluids (i.e. not superfluid or superconducting), we will use early Latin capital indices A, B etc., so a sum over all fluids (indexed by X, Y etc.) can be written as $\sum_X = \sum_A + \sum_\Upsilon$. Apart from the electric charge there seems to be no fundamental difference between superfluids and superconductors, and therefore we will in the following refer to them as "uncharged" and "charged superfluids" respectively. We note that the present treatment considers superfluids as a subclass of perfect fluids, and will therefore represent some restrictions as to the application to strongly anisotropic superfluid phases like they are found in ^{3}He [12], which is governed by additional "internal" degrees of freedom like the spin and angular momentum of the Cooper pairs. But at least for situations where these additional degrees of freedom of the order parameter can be considered as "frozen" and the dynamics mainly governed by the superfluid "phase" to be discussed in the following, the present approach should still represent an acceptable approximation.

We distinguish the (connected) spacetime domain $\mathcal{D}^\Upsilon$ occupied by the superfluid constituent Υ from the subset of its respective "superfluid domain" $\mathcal{S}^\Upsilon \subseteq \mathcal{D}^\Upsilon$, which corresponds to what is sometimes called the "bulk". In the superfluid domain $\mathcal{S}^\Upsilon$ the canonical momentum π^Υ_α always obeys the constraint

$$\pi^\Upsilon_\alpha = \hbar\nabla_\alpha\varphi^\Upsilon\,, \tag{17}$$

where the "phase" φ^Υ is a continuously differentiable scalar on $\mathcal{S}^\Upsilon$, that can be multi–valued, but the differences between values in the same point are restricted to be integer multiples of 2π. This reminds of an angle variable and reflects the role of φ^Υ as a quantum phase $e^{i\varphi}$. In addition to the property of (quantized) potential flow (17), the superfluid Υ in its superfluid domain $\mathcal{S}^\Upsilon$ is *perfectly inviscid*. In that sense a superfluid is probably the best representation of a perfect fluid in nature. On the other hand, outside its superfluid domain, i.e. in $D^\Upsilon \setminus \mathcal{S}^\Upsilon$, the superfluid is not constrained to potential flow (17) and can also possess some viscosity like a "normal" fluid. The property (17) implies that the canonical vorticity $w^\Upsilon_{\alpha\beta}$ vanishes on the whole superfluid domain $\mathcal{S}^\Upsilon$, i.e.

$$w^\Upsilon_{\alpha\beta} = 2\nabla_{[\alpha}\pi^\Upsilon_{\beta]} = 0\,, \tag{18}$$

which states that the superfluid is irrotational, and implies that the equation of motion (14) is automatically satisfied on $\mathcal{S}^\Upsilon$.

Irrotational flow is of course not restricted to superfluids, and the vortex–type configurations to be discussed later have been known long before the discovery of superfluids, familiar examples are tornados or water flowing out the drain of the bath tub. But the multi–valuedness of the "phase" of a perfect fluid in a state of potential flow is *not* subject to a "quantization" condition of integer multiples of 2π, and a perfect fluid only exists as an idealization of a "real" fluid with some nonvanishing amount of viscosity, contrary to the completely inviscid superfluids in the superfluid domain. Furthermore there is an important energy gain associated with the superfluid domain $\mathcal{S}^\Upsilon$, the so–called "condensation energy". Superfluids consequently try to maximize their superfluid domain $\mathcal{S}^\Upsilon$ (and thereby to satisfy (17)) as far as possible in the limits of the fluid domain $\mathcal{D}^\Upsilon$.

One of the most important consequences of (17) is that it allows for the topologically stable flow configurations known as vortices, which are characterized by the property that different values of the (multi–valued) phase φ^Υ in the same point can be connected by closed paths Γ that lie entirely in the superfluid domain $\mathcal{S}^\Upsilon$. As stated above, the difference can only be of the form $2\pi N^\Upsilon$, where the integer N^Υ is called the "winding number". The winding number N^Υ of a closed path $\Gamma \subset \mathcal{S}^\Upsilon$ can be written as

$$N^\Upsilon = \frac{1}{2\pi\hbar}\oint_\Gamma \pi^\Upsilon_\alpha ds^\alpha\,, \quad \Gamma \subset \mathcal{S}^\Upsilon\,. \tag{19}$$

It is evident that N^Υ does not change for continuously deformed paths $\Gamma \to \Gamma' \subset \mathcal{S}^\Upsilon$, and N^Υ is therefore a topological constant for each equivalence class of closed paths in $\mathcal{S}^\Upsilon$. A nonvanishing N^Υ implies that the path $\Gamma \subset \mathcal{S}^\Upsilon$ can not be continuously contracted to a point, because it would necessarily have to cross at least one point $P \notin \mathcal{S}^\Upsilon$ where the phase φ^Υ is not defined, and therefore $\mathcal{S}^\Upsilon$ is necessarily multiply connected if there are nonvanishing winding numbers N^Υ.

IV. THE "MONGREL" REPRESENTATION OF SUPERFLUID–NORMAL MIXTURES

In the previous section we have seen that a superfluid on its superfluid domain is generally characterized by a constraint (17) on the (canonical) superfluid momentum, while "normal" fluids are generally more easily described in terms of their particle number currents. For this reason it will turn out to be extremely convenient to pass from the "pure" type of representation used in (4), which expresses all the currents in terms of all the momenta (or vice–versa), to a "mongrel" representation where the *superfluid currents* and *normal momenta* are expressed in terms of the *superfluid momenta* and *normal currents*. This type of representation has for example been used tacitly as the base of Landau's two–fluid model for superfluid ^{4}He [13], which was formulated in terms of a "superfluid velocity", representing in fact the irrotational superfluid momentum of (17) (divided by a fixed mass), and of a "normal fluid" velocity, which represents the real mean velocity of the viscous gas of excitations. This will be seen in some more detail in the discussion of the two–fluid model in the concluding section VIII.

In order to pass to this mongrel representation, we decompose the entrainment matrix K_{XY} into a purely superfluid symmetric matrix $S_{\Upsilon\Psi}$, a symmetric matrix V_{AB} of purely normal ("viscous") fluids and a "mixed" superfluid–normal matrix $M_{\Upsilon A}$, so (4) can be written in this decomposition as

$$\boldsymbol{n}_\Upsilon = S_{\Upsilon\Psi}\,\boldsymbol{\mu}^\Psi + M_{\Upsilon B}\,\boldsymbol{\mu}^B\,, \tag{20}$$

$$\boldsymbol{n}_A = M_{A\Psi}\,\boldsymbol{\mu}^\Psi + V_{AB}\,\boldsymbol{\mu}^B\,, \tag{21}$$

where $M_{\Upsilon B} = M_{B\Upsilon}$. For clarity we use in this section **bold** typeset for denoting spacetime vectors and covectors, as the spacetime indices are not important here and can be put in any consistent way. Applying the inverse matrix V^{-1} to (21), we can easily rewrite these relations in the "mongrel" form

$$\boldsymbol{\mu}^A = -\mathbb{M}^A_\Psi\,\boldsymbol{\mu}^\Psi + \mathbb{V}^{AB}\boldsymbol{n}_B\,, \tag{22}$$

$$\boldsymbol{n}_\Upsilon = \mathbb{S}_{\Upsilon\Psi}\,\boldsymbol{\mu}^\Psi + \mathbb{M}^B_\Upsilon\boldsymbol{n}_B\,, \tag{23}$$

where we defined the new matrices

$$\begin{aligned} &\mathbb{V}^{AB} \equiv \left(V^{-1}\right)^{AB}\,, \quad \mathbb{M}^A_\Psi \equiv \mathbb{V}^{AB} M_{B\Psi}\,, \\ &\mathbb{S}_{\Upsilon\Psi} \equiv S_{\Upsilon\Psi} - M_{\Upsilon A}\mathbb{V}^{AB} M_{B\Psi}\,. \end{aligned} \tag{24}$$

In this representation it is easy to see that terms of the form $\boldsymbol{n}_X\,\boldsymbol{\mu}^X$, e.g. in the stress–energy tensor (16), can be written in the "quasi separated" form

$$\boldsymbol{n}_X\,\boldsymbol{\mu}^X = \boldsymbol{\mu}^\Upsilon\,\mathbb{S}_{\Upsilon\Psi}\,\boldsymbol{\mu}^\Psi + \boldsymbol{n}_A\,\mathbb{V}^{AB}\,\boldsymbol{n}_B\,, \tag{25}$$

where the effect of "mixed" entrainment between superfluids and normal fluids is hidden in the use of the matrix $\mathbb{S}$. As we consistently wrote lower constituent indices for currents and upper constituent indices for momenta, we can now use this convention to introduce a very convenient and suggestive notation, namely to use $\mathbb{S}_{\Upsilon\Psi}$ to *lower* superfluid indices Υ, Ψ etc., and $\mathbb{V}^{AB}$ to *raise* normal fluid indices A, B etc. This can formally be understood as choosing $\mathbb{S}$ and $\mathbb{V}$ as the *metric tensors* in the respective constituent vector spaces of the superfluids and the normal fluids, but can also just be seen as a shorthand notation for

$$\boldsymbol{\mu}_\Upsilon \equiv \mathbb{S}_{\Upsilon\Psi}\,\boldsymbol{\mu}^\Psi\,, \quad \text{and} \quad \boldsymbol{n}^A \equiv \mathbb{V}^{AB}\,\boldsymbol{n}_B\,. \tag{26}$$

In this notation, stress–energy contributions $\boldsymbol{n}_X\,\boldsymbol{\mu}^X$ take the simple and concise form

$$\boldsymbol{n}_X\,\boldsymbol{\mu}^X = \boldsymbol{n}^A\,\boldsymbol{n}_A + \boldsymbol{\mu}^\Upsilon\,\boldsymbol{\mu}_\Upsilon\,, \tag{27}$$

where all the information about entrainment has been encoded in the respective metrics of the superfluid and normal constituent spaces.

We note that the superfluid constraint (17) generally applies to the *canonical* momenta $\boldsymbol{\pi}^\Upsilon$, which only in the case of uncharged superfluids coincide with the dynamical momenta $\boldsymbol{\mu}^\Upsilon = \boldsymbol{\pi}^\Upsilon - e^\Upsilon \boldsymbol{A}$. This implies a qualitative difference between charged and uncharged superfluids, and it will be useful to separate the superfluid constituent space into the two orthogonal subspaces that are naturally defined by the superfluid "charge vector" with components e^Υ. The respective subspaces are defined by parallel and orthogonal projection via the projection tensors

$$\eta^\Upsilon_\Psi \equiv \frac{e^\Upsilon e_\Psi}{(e^\Lambda e_\Lambda)}\,, \quad \gamma^\Upsilon_\Psi \equiv \delta^\Upsilon_\Psi - \eta^\Upsilon_\Psi\,, \tag{28}$$

where again we have used the notation $e_\Upsilon \equiv \mathbb{S}_{\Upsilon\Psi} e^\Psi$. Now we can decompose constituent vectors, e.g. the superfluid momenta as $\boldsymbol{\mu}^\Upsilon = \boldsymbol{\mu}^\Upsilon_\parallel + \boldsymbol{\mu}^\Upsilon_\perp$, where

$$\boldsymbol{\mu}^\Upsilon_\parallel \equiv \eta^\Upsilon_\Psi \boldsymbol{\mu}^\Psi\,, \quad \text{and} \quad \boldsymbol{\mu}^\Upsilon_\perp \equiv \gamma^\Upsilon_\Psi \boldsymbol{\mu}^\Psi\,. \tag{29}$$

The subtlety of this notation is that even though a "parallel" constituent vector $\boldsymbol{\mu}^\Upsilon_\parallel$ only has nonvanishing components for charged superfluid constituents, and respectively $\boldsymbol{\mu}^\Upsilon_\perp$ only for uncharged superfluids, the *values* of the respective components may depend on all the other superfluids *and* normal fluids, as the projection tensors contain the entrainment matrix $\mathbb{S}$.

V. THE STATIONARY CYLINDRICAL VORTEX CONFIGURATION

In this work we will consider the simplest, because maximally symmetric type of vortex configuration, which is characterized by both stationarity and cylindrical symmetry. This means that there are three independent, commuting (in the sense of Lie brackets) symmetry generators k^α, l^α and m^α, which can be taken to correspond to time translations, longitudinal space translations (along the vortex axis) and axial rotations, respectively. The geometric picture of the symmetry surfaces generated by k^α, l^α and m^α are cylindrical hypersurfaces that build a well behaved foliation of spacetime, and can therefore be parametrized by a "radial" coordinate r. Let us introduce the corresponding cylindrical coordinates $\{x^0, x^1, x^2, x^3\} = \{t, z, \varphi, r\}$, adapted to these symmetries, i.e.

$$k^\alpha = \{1,0,0,0\}\,, \quad l^\alpha = \{0,1,0,0\}\,, \quad m^\alpha = \{0,0,1,0\}\,. \tag{30}$$

The symmetry requirements and the property of conserved currents (14), i.e. $\nabla_\alpha n^\alpha_X = 0$, restrict the flow to be purely helical, i.e., to have no radial components. Therefore the currents are confined to timelike hypersurfaces generated by the symmetry vectors and can be written as

$$n^\alpha_X = \{\, n^t_X(r),\, n^z_X(r),\, n^\varphi_X(r),\, 0\,\}\,. \tag{31}$$

A further consequence of the symmetry is that any physically well defined quantity Q of the flow must be invariant under symmetry translations, which means that the corresponding Lie derivatives must vanish, i.e. $\mathcal{L}_\xi Q = 0$, for ξ^α being any linear combination (with constant coefficients) of the symmetry vectors k^α, m^α and l^α. This also holds for gauge dependent quantities like the canonical momentum π^X_α, provided we fix the gauge in a way that respects the same symmetries, i.e. when $(\mathcal{L}_\xi A)_\alpha = 0$. Such a gauge choice is given by

$$A_\alpha = \{\, A_t(r),\, A_z(r),\, A_\varphi(r),\, 0\,\}\,. \tag{32}$$

The components A_t and A_z are still subject to the residual gauge freedom of an additive constant, i.e.

$$A_t \to A_t + \mathcal{G}_t\,, \quad A_z \to A_z + \mathcal{G}_z\,, \tag{33}$$

but because φ is an angle variable, corresponding to a compact dimension, the gauge of the axial component A_φ is completely fixed by (32). This is most easily seen by applying Stoke's theorem to a $\{r,\,\varphi\}$–surface integral over $F_{\alpha\beta}$, i.e. $\int d\Sigma^{\alpha\beta} F_{\alpha\beta} = \oint dl^\alpha A_\alpha$, which in this trivial symmetric case just reduces to $\int_0^{r_\infty} dr\,(dA_\varphi/dr) = A_\varphi(r_\infty)$, and so the gauge is fixed as

$$A_\varphi(0) = 0\,. \tag{34}$$

With the gauge choice (32), the symmetry condition for π^X_α reads

$$\left(\mathcal{L}_\xi \pi^X\right)_\alpha = 0\,, \tag{35}$$

where ξ^α can be any linear combination of the three symmetry generators. The well known Cartan formula for the Lie derivative of a p–form $w_{\alpha\beta\gamma\ldots}$, namely

$$(\mathcal{L}_\xi\, w)_{\alpha\beta\gamma\ldots} = (p+1)\,\xi^\lambda \nabla_{[\lambda} w_{\alpha\beta\gamma\ldots]} + p\,\nabla_{[\alpha}\left(\xi^\lambda w_{\lambda\beta\gamma\ldots]}\right)\,, \tag{36}$$

can be applied to the 1–form π^X_α in (35), and so we obtain the explicit symmetry condition,

$$2\xi^\beta\,\nabla_{[\beta}\pi^X_{\alpha]} + \nabla_\alpha\left(\xi^\beta\pi^X_\beta\right) = 0\,. \tag{37}$$

For superfluids (in the superfluid domain), the first term vanishes because of the irrotationality property (18), and so the second term provides us with three independent integrals of motion, corresponding to the three symmetry generators, namely

$$-E^\Upsilon \equiv k^\alpha\pi^\Upsilon_\alpha\,,\quad L^\Upsilon \equiv l^\alpha\pi^\Upsilon_\alpha\,,\quad M^\Upsilon \equiv m^\alpha\pi^\Upsilon_\alpha\,, \tag{38}$$

interpretable respectively as the *energy*, (canonical) *longitudinal momentum*, and (canonical) *angular momentum* per particle. While E^Υ and L^Υ are generally subject to the residual gauge freedom (33) of an additive constant (except in the uncharged cases $e^\Upsilon = 0$), the axial constant M^Υ is *not*, because there is no gauge freedom for A_φ. In order to calculate the winding numbers N^Υ of the vortex by (19), we have to choose a path Γ enclosing the vortex axis. Such a path can always be continuously deformed into a path generated by m^α alone, and so by (38) the integration simply yields

$$N^\Upsilon = \frac{M^\Upsilon}{\hbar}\,. \tag{39}$$

Therefore the constant (canonical) angular momentum per particle, M^Υ, is an integer multiple of $\hbar$, the fundamental quantum of angular momentum, and the corresponding angular momentum "quantum number" is just the winding number N^Υ. The superfluid canonical momenta $\boldsymbol{\pi}^\Upsilon = \boldsymbol{\mu}^\Upsilon + e^\Upsilon\boldsymbol{A}$ are thereby completely determined (in the superfluid domain) by the integrals of motion (38) (modulo the gauge freedom (33)), namely

$$\pi^\Upsilon_\alpha = \{-E^\Upsilon, L^\Upsilon, \hbar N^\Upsilon, 0\}\,,\quad \text{with}\quad N^\Upsilon \in \mathbb{Z}\,, \tag{40}$$

where the vanishing of the radial component π^Υ_r follows from the helical direction (31) of the currents n^α_X, and the entrainment relation (4) together with (11) and the gauge choice (32).

In a more realistic treatment, the normal fluids are expected to have some amount of viscosity, in which case the condition of stationarity, which excludes all dissipative motion, restricts all the normal currents to be comoving with the same uniform rotation Ω, i.e.

$$n^\alpha_A = n^t_A\,v^\alpha\,,\quad \text{with}\quad v^\alpha \equiv k^\alpha + \Omega m^\alpha = \{1, 0, \Omega, 0\}\,. \tag{41}$$

We could also have allowed for a constant longitudinal velocity along l^α, but this is trivially annihilated by a Lorentz boost, and so we have chosen our reference frame at rest with respect to the longitudinal motion of the normal fluids. The symmetry condition (35) along the flowlines of the normal fluids, i.e. with $\xi^\alpha = v^\alpha$, together with the equation of motion (14) yields one integral of motion for each normal fluid, namely

$$-\bar{E}^A = v^\alpha\pi^A_\alpha\,. \tag{42}$$

With the given restrictions on the currents (31) and (41), the integrals of motion E^Υ, L^Υ, N^Υ, $\bar{E}^A$ and Ω are sufficient for the equations of motion (14) to be satisfied. But in order to actually integrate these differential equations, one is still left with the generally nontrivial problem of solving equations for the spacetime metric $g_{\alpha\beta}$, together with Maxwell's equation (12) for the gauge field A_α. However, for most vortex applications of practical interest (including those in neutron stars), the gravitational self–interaction of the vortex can be completely neglected, so the background metric can in any case be considered as given in advance. Furthermore, as the radial dimensions of vortices are generally much smaller than the lengthscale of gravitational curvature, the local spacetime metric of the vortex can safely be considered as flat, and so in cylindrical coordinates we can write it as

$$ds^2 \equiv g_{\alpha\beta}\,dx^\alpha\,dx^\beta = -dt^2 + dz^2 + r^2\,d\varphi^2 + dr^2\,. \tag{43}$$

The remaining differential equation to be solved is (12) for the electromagnetic gauge field A_α. The necessary coefficients of the metric connection can easily be calculated for the flat metric (43), and we find the explicit Maxwell equations for the gauge field A_α in the form

$$(rA_t')' = 4\pi r j^t\,, \quad -(rA_z')' = 4\pi r j^z\,, \tag{44}$$

$$-\left(\frac{A_\varphi'}{r}\right)' = 4\pi r j^\varphi\,, \tag{45}$$

where the prime denotes differentiation with respect to r. Equations (44) describe a radial electric field A_t' created by the charge distribution j^t, and an axial magnetic field A_z' around a longitudinal current j^z. These equations will result in exponentially "screened" solutions, typical of charged superfluids. As we saw in section III, the vortex is characterized by nonvanishing winding numbers N^Υ, which by (11) and (4) are seen to be directly related to the axial components j^φ and will result in a screened longitudinal magnetic field B_z, which is conventionally defined as

$$B_z \equiv \frac{A_\varphi'}{r}\,. \tag{46}$$

VI. REFERENCE STATE AND VORTEX PROPERTIES

A. The reference state

In the previous section we have completely specified the fluid configuration containing a vortex, but in order to separate the quantities attributed to the vortex from the fluid "background", we first have to specify this reference "background" state, which will be denoted by the subscript $\ominus$. For any quantity Q, the part $\delta_\ominus Q$ attributed to the vortex is defined as the difference with respect to the corresponding reference value $Q_\ominus$, i.e.

$$\delta_\ominus Q \equiv Q - Q_\ominus\,. \tag{47}$$

The reference state should respect at least the same symmetries as the vortex state, and can therefore, by the reasoning in Sec. V, be characterized completely by constants $E^\Upsilon_\ominus$, $L^\Upsilon_\ominus$, $N^\Upsilon_\ominus$, $\bar{E}^A_\ominus$ and $\Omega_\ominus$. Furthermore, we naturally want the the reference background to be "vortex free", which means that the topological constants characterizing a vortex have to vanish, i.e. $N^\Upsilon_\ominus = 0$. Another natural prescription is that the uniform rotation of the normal fluids should be the same in the reference state as in the vortex state, i.e. $\Omega_\ominus = \Omega$. However, there is no such "natural" choice for the remaining constants $E^\Upsilon_\ominus$, $L^\Upsilon_\ominus$ and $\bar{E}^A_\ominus$, if one allows for compressibility of the fluids. The compressibility is described by the fact that the entrainment matrix (5) is in general a function of the momentum scalars, i.e. $K_{XY} = K_{XY}(\mu^V_\alpha \mu^{W\alpha})$, and therefore, if $(\mu^V_\alpha \mu^{W\alpha})_\ominus \neq \mu^V_\alpha \mu^{W\alpha}$, this generally entails that $K_{XY} \neq K^\ominus_{XY}$. Now, if we consider for example the t component of the relation (4) between currents and momenta, and if for illustration we suppose for a moment that there are no normal fluids, then $n^t_\Upsilon = K_{\Upsilon\Psi}\mu^{\Psi t}$, and $n^t_{\Upsilon\ominus} = K^\ominus_{\Upsilon\Psi}\mu^{\Psi t}_\ominus$. Choosing for example the straightforward reference constants $E^\Upsilon_\ominus = E^\Upsilon$ and $L^\Upsilon_\ominus = L^\Upsilon$, leads to changed particle densities $n^t_{\Upsilon\ominus} \neq n^t_\Upsilon$, and especially changed *mean* particle number densities (in the region of integration with the upper cutoff radius r_∞), i.e. $\overline{n^t_{\Upsilon\ominus}} \neq \overline{n^t_\Upsilon}$. We see that with this choice of reference constants, we compare a vortex state with a reference state that does not have the same number of particles in the region of integration. Another physically interesting choice of reference state would therefore rather consist in readjusting the reference constants $E^\Upsilon_\ominus$ in such a way as to obtain the same *mean* particle number densities (and therefore total number of particles in the region of integration) in the reference state. These different choices have been analyzed and properly accounted for in [14] for the case of a vortex in an uncharged superfluid, and are found to be inequivalent to each other, even in the limit $r_\infty \to \infty$.

Due to the additional complications of multiple entrainment and charged fluids in the present analysis, we will postpone this problem of compressibility effects to future work, and restrict our attention here to the simpler case of a "stiff" equation of state that is characterized by a constant entrainment matrix, i.e.

$$\frac{\partial K_{XY}}{\partial(\mu^V_\alpha \mu^{W\alpha})} = 0 \quad \Longrightarrow \quad K^\ominus_{XY} = K_{XY}\,. \tag{48}$$

In this "stiff" case, the most natural reference state is unambiguously characterized just by choosing the longitudinal superfluid momentum components $E^\Upsilon_\ominus$, $L^\Upsilon_\ominus$ to be the same as in the vortex state, i.e.

$$\pi^{\Upsilon\ominus}_\alpha \equiv \{-E^\Upsilon, L^\Upsilon, 0, 0\}\,, \tag{49}$$

while the constants $\bar{E}^A$ can be fixed by taking the normal particle densities to be unchanged with respect to the vortex state, i.e.

$$n^\alpha_{A\ominus} \equiv n^t_A v^\alpha \,, \quad \text{where} \quad v^\alpha = \{1, 0, \Omega, 0\} \,. \tag{50}$$

Due to the assumption of a stiff equation of state (48), all longitudinal current components n^t_X and n^z_X remain unchanged in the reference state. Furthermore we will assume the electric current to vanish in the reference state, i.e.

$$j^\alpha_\ominus = 0 \,, \tag{51}$$

which implies that the longitudinal electric current also vanishes in the vortex state,

$$j^t = j^t_\ominus = 0 \,, \quad \text{and} \quad j^z = j^z_\ominus = 0 \,, \tag{52}$$

and so we also have from (44) (in an appropriate gauge)

$$A_t = A^\ominus_t = 0 \,, \quad \text{and} \quad A_z = A^\ominus_z = 0 \,. \tag{53}$$

The reference state is now completely fixed by the properties (49), (50) and (51). The vortex modifies only the φ components of currents and momenta, so it will be convenient to introduce for covectors Q_α the short notation $\widetilde{Q} \equiv \delta_\ominus Q_\varphi$ for the part of the Q_φ that is due to the vortex, and $Q_\ominus \equiv Q^\ominus_\varphi$ for the part that is still present in the reference state, e.g.

$$\mu^\Upsilon_\varphi = \widetilde{\mu}^\Upsilon + \mu^\Upsilon_\ominus \,, \quad \text{and} \quad A_\varphi = \widetilde{A} + A_\ominus \,. \tag{54}$$

From (40) and (49) it is easy to see that

$$\widetilde{\mu}^\Upsilon = \hbar N^\Upsilon - e^\Upsilon \widetilde{A} \,, \quad \text{and} \quad \mu^\Upsilon_\ominus = -e^\Upsilon A_\ominus \,. \tag{55}$$

B. The London field

Contrary to the longitudinal components $A^\ominus_t$ and $A^\ominus_z$, the axial gauge field $A_\ominus$ in the reference state will not be trivial, due to the uniform rotation of the charged normal fluids. The Maxwell equation (45) for the φ component in the reference state, i.e. $(A'_\ominus/r)' = 0$, allows for a uniform magnetic field $B_\ominus$ in z direction (defined as in (46)), namely by integration and using (34) one gets,

$$B_\ominus \equiv \frac{A'_\ominus}{r} = \frac{2}{r^2} A_\ominus = \text{const.} \,, \tag{56}$$

where $B_\ominus$ is in fact the well known uniform London field of rotating superconductors. An explicit expression for the London gauge field $A_\ominus$ can be obtained simply from the reference property $j^\varphi_\ominus = e^X n^\varphi_{X\ominus} = 0$, together with the "mongrel" entrainment expression (23), and relation (55), which yields

$$A_\ominus = r^2 \left(e^\Psi e_\Psi\right)^{-1} \left(e^A + e^\Upsilon \mathbb{M}^A_\Upsilon\right) n^\varphi_A \,, \tag{57}$$

and after using (41) to write $n^\varphi_A = \Omega\, n^t_A$, we get the London field $B_\ominus$ as

$$B_\ominus = 2\Omega \left(e^\Psi e_\Psi\right)^{-1} \left(e^A + e^\Upsilon \mathbb{M}^A_\Upsilon\right) n^t_A \,. \tag{58}$$

The London field $B_\ominus$ is seen to be proportional to the uniform rotation Ω of the normal fluids. If we now use the additional property of the vanishing charge density (51) in the reference state, i.e. $j^t_\ominus = 0$, then we can finally obtain the very simple expression for the London field,

$$B_\ominus = -2\Omega \, \frac{e^\Upsilon \mathbb{S}_{\Upsilon\Psi} E^\Psi}{e^A \mathbb{S}_{A\Theta} e^\Theta} = -2\Omega \frac{e_\Upsilon E^\Upsilon}{e_\Psi e^\Psi} \,, \tag{59}$$

where we have used the notation of lowering and raising constituent indices via the matrix $\mathbb{S}$ introduced in Sec. IV. If we consider in particular the case of a single charged superfluid with mass per particle m and charge per particle e, this expression in the Newtonian limit, where $E^\Upsilon \to m$, reduces to the well known expression $B_\ominus = -2\Omega m/e$. The question of whether m in this formula should represent the bare mass or some "effective" mass per particle will be discussed briefly in the concluding section VIII.

C. The Magnetic field of the vortex

The reference state properties (49) and (50) further allow us to rewrite the axial current j^φ in the form $j^\varphi = e^\Upsilon(n^\varphi_\Upsilon - n^\varphi_{\Upsilon\ominus})$, and with (23) we obtain the compact form

$$j^\varphi = \frac{1}{r^2} e^\Upsilon \, \mathbb{S}_{\Upsilon\Psi} \, \widetilde{\mu}^\Psi = \frac{1}{r^2} e_\Upsilon \, \widetilde{\mu}^\Upsilon \, . \tag{60}$$

Inserting this into the corresponding Maxwell equation (45) gives

$$e^\Upsilon \, \mathbb{S}_{\Upsilon\Psi} \, \widetilde{\mu}^\Psi = -\frac{r}{4\pi} \widetilde{B}' \, , \tag{61}$$

which can be written more explicitly as a differential equation for $\widetilde{A}$, containing the winding numbers N^Υ as parameters, namely

$$(e^\Psi e_\Psi) \widetilde{A} = \hbar \, (e_\Psi N^\Psi) + \frac{r}{4\pi} \widetilde{B}' \, , \tag{62}$$

where the longitudinal magnetic field of the vortex, $\widetilde{B} = \delta_\ominus B_z$, is defined following (46) as $\widetilde{B}(r) \equiv \widetilde{A}'(r)/r$. This second order differential equation for $\widetilde{A}$ (or $\widetilde{B}$) is of the modified Bessel type, and the asymptotic behavior of the solutions in the limit $r \to \infty$ can be derived directly from this equation, namely (where "$\sim$" means asymptotically proportional)

$$\begin{aligned} \widetilde{B} \sim \widetilde{B}' &\sim e^{-r/\ell} \, , \\ \lim_{r\to\infty} \widetilde{A} &= \hbar \, \frac{e_\Psi N^\Psi}{e^\Upsilon e_\Upsilon} \, , \end{aligned} \tag{63}$$

where ℓ is the so–called London penetration depth, which is given by the expression

$$\ell^{-2} \equiv 4\pi \, e^\Psi e_\Psi \, . \tag{64}$$

In the Newtonian limit of a single superfluid with charge per particle e, mass per particle m and a particle number density n, the matrix $\mathbb{S}$ reduces to n/m, and (64) reduces to the standard expression $\ell^{-2} = 4\pi e^2 n/m$.

The total electromagnetic flux of the vortex, $\Phi \equiv \oint \widetilde{A}_\alpha dx^\alpha$, for a circuit at sufficiently large radial distance, is easily seen from (63) to be given as

$$\Phi = 2\pi\hbar \, \frac{e_\Psi N^\Psi}{e^\Upsilon e_\Upsilon} \, , \tag{65}$$

which again reduces to the standard expression $\Phi = N(2\pi\hbar/e)$ in the Newtonian limit of a single charged superfluid with charge per particle e. The explicit solution of equation (62) is expressible in terms of the (modified) Bessel functions K_0 and K_1, namely

$$\begin{aligned} \widetilde{B}(r) &= C_0 \, K_0(r/\ell) \, , \\ \widetilde{A}(r) &= \frac{\Phi}{2\pi} - C_0 \, r\ell \, K_1(r/\ell) \, . \end{aligned} \tag{66}$$

This solution is only valid in the "common superfluid domain", i.e. in $\bigcap_\Upsilon \mathcal{S}^\Upsilon$, where all the constant winding numbers N^Υ are defined. From the divergence of $\widetilde{B}(r)$ on the axis it is evident that the common superfluid domain must have a finite separation, ξ say, from the axis, which can be used to define what is usually called the "vortex core", with ξ being the "core radius". The constant of integration C_0 is to be determined from the matching of (66) with the "inner" vortex solution, i.e. for $r \le \xi$. By integrating (66) for $r \ge \xi$, we get the vortex flux outside the core, i.e. $\Phi - \Phi_{\rm core}$, and so C_0 can be expressed in terms of the quantities ξ and the core magnetic flux $\Phi_{\rm core}$, namely

$$C_0 = \frac{\Phi - \Phi_{\rm core}}{2\pi\ell^2 x_0 K_1(x_0)} \, , \tag{67}$$

where x_0 is the rescaled core radius, $x_0 \equiv \xi/\ell$, which corresponds to the inverse of the Ginzburg–Landau parameter $\kappa \equiv \ell/\xi$ of the Ginzburg–Landau model. The limit of an extreme type–II superconductor is characterized by $\kappa \to \infty$, i.e. $x_0 \to 0$, $x_0 K_1(x_0) \to 1$, so the core structure becomes negligible, $\Phi_{\rm core} \ll \Phi$, and we get

$$C_0 \simeq \frac{\Phi}{2\pi\ell^2} = 4\pi\hbar \, e_\Psi N^\Psi \, , \quad \text{for} \quad \ell \gg \xi \, . \tag{68}$$

VII. THE VORTEX ENERGY

In this section we will consider the "macroscopic" properties of the vortex, namely its total energy per unit length and the tension of the vortex line. These quantities are obtained by integrating the local stress–energy tensor of the vortex, $\delta_\ominus T^\alpha{}_\beta$, over the spatial section $\{r,\varphi\}$ orthogonal to the ("longitudinal") vortex symmetry axes, whose coordinates are the subset $\{x^i\}=\{t,z\}$, for $\{i\}=\{0,1\}$. The local stress–energy coefficients of the vortex are seen from (16) to have the form

$$\delta_\ominus T^\alpha{}_\beta = \delta_\ominus\left(n^\alpha_X \mu^X_\beta\right) + \frac{1}{4\pi}\delta_\ominus\left(F^{\alpha\lambda}F_{\beta\lambda}\right) + \left[\delta_\ominus \mathcal{P} - \frac{1}{16\pi}\delta_\ominus\left(F^{\rho\lambda}F_{\rho\lambda}\right)\right] g^\alpha{}_\beta\,. \tag{69}$$

The "sectional" $\{r,\varphi\}$–integral is only meaningful for quantities that are scalars with respect to the sectional coordinates r and φ, and so we have to consider only the "longitudinally" projected tensor $\delta_\ominus T^i{}_j$. Another "sectional" scalar of the stress–energy tensor is the trace of the orthogonally projected components, which defines the local lateral pressure Π of the vortex,

$$2\Pi \equiv \delta_\ominus\left(T^\alpha{}_\alpha - T^i{}_i\right)\,. \tag{70}$$

In the case of a "stiff" equation of state (48), the Taylor expansion of $\mathcal{P}(\mu^X_\alpha \mu^{Y\alpha})$ around the reference state value $\mathcal{P}_\ominus \equiv \mathcal{P}((\mu^X_\alpha\mu^{Y\alpha})_\ominus)$ has only two terms (using (5)), namely

$$\mathcal{P}(\mu^X_\alpha \mu^{Y\alpha}) = \mathcal{P}_\ominus - \frac{1}{2}K_{XY}\,\delta_\ominus(\mu^X_\alpha\mu^{Y\alpha})\,. \tag{71}$$

The mongrel representation (Sec. IV) is particularly convenient to evaluate contributions of this type, because by the reference property (50) we have $\delta_\ominus(\boldsymbol{n}_A \mathbb{V}^{AB}\boldsymbol{n}_B)=0$, and so we find, using (4) and (25),

$$K_{XY}\,\delta_\ominus(\boldsymbol{\mu}^X\boldsymbol{\mu}^Y) = \delta_\ominus(\boldsymbol{n}_X\boldsymbol{\mu}^X) = \mathbb{S}_{\Upsilon\Psi}\,\delta_\ominus(\boldsymbol{\mu}^\Upsilon\boldsymbol{\mu}^\Psi)\,. \tag{72}$$

The relevant contributions (69) of $\delta_\ominus T^\alpha{}_\beta$ are now straightforward to evaluate, and are found to be given by

$$\delta_\ominus\left(n^i_X\mu^X_j\right)=0\,,\quad \delta_\ominus\left(F^{i\lambda}F_{j\lambda}\right)=0\,, \tag{73}$$

$$\delta_\ominus\left(F^{\alpha\beta}F_{\alpha\beta}\right) = 2\widetilde{B}^2 + 4\widetilde{B}B_\ominus\,, \tag{74}$$

$$\delta_\ominus T^\alpha{}_\alpha = -\delta_\ominus\left(n^\alpha_X\mu^X_\alpha\right) = 2\delta_\ominus\mathcal{P}\,, \tag{75}$$

$$\delta_\ominus\mathcal{P} = -\frac{1}{2r^2}\mathbb{S}_{\Upsilon\Psi}\left(\widetilde{\mu}^\Upsilon\,\widetilde{\mu}^\Psi + 2\widetilde{\mu}^\Upsilon\,\mu^\Psi_\ominus\right)\,. \tag{76}$$

Putting these results into the expression for the vortex stress–energy tensor (69), we find that the longitudinally projected tensor $\delta_\ominus T^i{}_j$ is proportional to the unit tensor, i.e

$$\delta_\ominus T^i{}_j = -\widetilde{T} g^i{}_j\,, \tag{77}$$

with

$$\widetilde{T} = \frac{1}{16\pi}\delta_\ominus\left(F^{\alpha\beta}F_{\alpha\beta}\right) - \delta_\ominus\mathcal{P}\,, \tag{78}$$

and so the vortex energy density, $\delta_\ominus T^{00}$, is equal to the (local) longitudinal tension of the vortex, $-\delta_\ominus T^{zz}$, a property that is characteristic of the stiff equation of state (48). The vortex energy per unit length U is defined as the sectional integral

$$U \equiv -2\pi\int_0^{r_\infty} dr\, r\,\delta_\ominus T^0{}_0 = 2\pi\int_0^{r_\infty} dr\, r\,\widetilde{T}\,. \tag{79}$$

The energy density $\widetilde{T}$ can be decomposed into two parts,

$$\widetilde{T} = \widetilde{T}_{\rm vort} + \widetilde{T}_{\rm rot}, \tag{80}$$

where $\widetilde{T}_{\rm vort}$ is the part that is independent of the rotation Ω of the normal fluids,

$$\widetilde{T}_{\rm vort} = \frac{1}{2r^2}\widetilde{\mu}_\Psi \widetilde{\mu}^\Psi + \frac{1}{8\pi}\widetilde{B}^2\,, \tag{81}$$

while $\widetilde{T}_{\rm rot}$ is proportional to Ω (via $B_\ominus$, see equ. (59)),

$$\widetilde{T}_{\rm rot} = B_\ominus \left(\frac{1}{4\pi}\widetilde{B} - \frac{1}{2}e_\Psi \widetilde{\mu}^\Psi\right)\,, \tag{82}$$

and the lateral pressure Π, defined in (70), is found to be given by

$$\Pi = \frac{1}{8\pi}\left[\widetilde{B}^2 + 2\widetilde{B}B_\ominus\right]\,. \tag{83}$$

Expression (81) for $\widetilde{T}_{\rm vort}$ can be transformed using Maxwell's equation (61) into the "nearly integrated" form

$$\widetilde{T}_{\rm vort} = \frac{\hbar^2}{2r^2}\left[N^\Psi N_\Psi - \frac{(e_\Upsilon N^\Upsilon)^2}{e^\Psi e_\Psi}\right] - \frac{e_\Upsilon N^\Upsilon}{e_\Psi e^\Psi}\frac{\hbar}{8\pi r}\widetilde{B}' + \frac{1}{8\pi r}\left(\widetilde{A}\widetilde{B}\right)'\,. \tag{84}$$

The easiest way to see this is to first expand only one $\widetilde{\mu}^\Upsilon$ in (81) using (55) and apply (61), then expand the remaining $\widetilde{\mu}^\Psi$ and use the second form of Maxwell's equation (62) for $\widetilde{A}$. In order to regroup the derivatives, one also has to expand one $\widetilde{B}$ as $\widetilde{A}'/r$ in the last term of (81). In a similar way, $\widetilde{T}_{\rm rot}$ can be reduced to

$$\widetilde{T}_{\rm rot} = \frac{B_\ominus}{8\pi r}\left(r^2\widetilde{B}\right)'\,. \tag{85}$$

As anticipated from the divergence of the magnetic field (66) on the vortex axis, we encounter the same problem in the energy density (84). This well known fact is due to the constant superfluid (canonical) angular momentum per particle, $\pi^\Upsilon_\varphi = \hbar N^\Upsilon$, in the superfluid domain $\mathcal{S}^\Upsilon$. Therefore each superfluid with a nonvanishing winding number $N^\Upsilon \neq 0$, must have some finite "core" region separating the respective superfluid domain $\mathcal{S}^\Upsilon$ from the vortex axis. The actual size of the respective core region is determined by a trade–off between the loss of condensation energy associated with the core region, and the diverging energy density (84) in the superfluid domain. The detailed description of this superfluid–normal transition would ask for either a microscopic theory, or at least some phenomenological, e.g. Ginzburg–Landau type description of the involved superfluids. However, such detailed descriptions turn out to be unnecessary for our present purpose, as we can proceed on the basis of a very general hydrodynamic description of the vortex core, based only on the necessary "minimal assumptions" needed to avoid the energy divergence. Namely, as the superfluid constraint (17) does no longer apply in the respective "core" regions, the (canonical) angular momentum π^Υ_φ there is not quantized, and is allowed to depend on the radial variable r. The winding number N^Υ is strictly speaking not defined in the core region, but we can keep the same symbol as a shorthand notation for $\pi^\Upsilon_\varphi/\hbar$, so we cast our general description of the core region in the simple form

$$N^\Upsilon(r) = \begin{cases} N^\Upsilon \in \mathbb{Z} & \text{for} \quad r > \xi \\ \mathcal{N}^\Upsilon(r) & \text{for} \quad r \leq \xi \end{cases}\,, \tag{86}$$

where $\mathcal{N}^\Upsilon(r)$ is a continuous, monotonic function, which has to ensure the vortex energy density $\widetilde{T}$ to remain finite on the vortex axis, i.e. in the limit $r \to 0$. Note that the "core radius" ξ is defined, as in Sec. V, as the radial distance of the "common superfluid domain" $\bigcap_\Upsilon \mathcal{S}^\Upsilon$ for the vortex axis, and is therefore the maximum core radius of the individual superfluids. This obviously does not restrict the generality of the core description (86), as the $\mathcal{N}^\Upsilon(r)$ are allowed to remain constant until some smaller radius $\xi^\Upsilon < \xi$. In order to have a regular behavior of the energy density $\widetilde{T}$ near the axis, it is sufficient to demand that $\mathcal{N}(r)$ and $\widetilde{A}(r)$ vanish on the vortex axis *at least* as

$$\mathcal{N}(r) \sim r\,, \quad \text{and} \quad \widetilde{A} \sim r^2 \quad \text{for} \quad r \to 0\,, \tag{87}$$

where by "$\sim$" we mean "asymptotically proportional" (and not necessarily equal). This phenomenological description is based on only two parameters, the "core radius" ξ and the core condensation energy per unit length $U_{\rm con}$. These two phenomenological parameters would have to be determined either from experiment or from a microscopic theory, but the model is now sufficiently determined to allow the integration of the vortex energy, without the need of further assumptions concerning the underlying physical processes of superfluidity.

The total vortex energy per unit length is

$$U = U_{\rm con} + U_{\rm vort} + U_{\rm rot}\,, \tag{88}$$

where according to (79) and (80) we have defined

$$\begin{aligned} U_{\rm vort} &\equiv 2\pi \int_0^{r_\infty} dr\, r\, \widetilde{T}_{\rm vort}\,, \\ U_{\rm rot} &\equiv 2\pi \int_0^{r_\infty} dr\, r\, \widetilde{T}_{\rm rot}\,. \end{aligned} \tag{89}$$

The energy contribution $U_{\rm rot}$, which is proportional to the rotation Ω of the normal fluids, is found from (85) to be

$$U_{\rm rot} = \frac{B_\ominus}{4} \left(r^2 \widetilde{B}\right)\Big|_0^{r_\infty} = 0\,, \tag{90}$$

where the vanishing of the integral follows from the asymptotic properties (63) and (87) of the magnetic field $\widetilde{B}$. In the Newtonian description of a rotating superconductor, the vortex energy was already found [9] to be unchanged by the rotating charged background, and this lemma is seen here to still hold under quite general conditions:

Rotation energy cancellation lemma: *The "hydrodynamic" energy per unit length (i.e. excluding the core condensation energy $U_{\rm con}$) of a cylindrically symmetric and stationary vortex in a "stiff" mixture of interacting superfluids, superconductors and normal fluids (48) is independent of the uniform rotation rate Ω of the normal fluids, despite the fact that the radial distribution of the hydrodynamic energy density is modified by Ω, as seen in (85).*

The vortex energy contribution $U_{\rm vort}$ in (89) is found by integrating (84), which yields

$$\begin{aligned} U_{\rm vort} = \pi\hbar^2 &\left[N^\Psi N_\Psi - \frac{(e_\Upsilon N^\Upsilon)^2}{e_\Psi e^\Psi}\right] \ln\frac{r_\infty}{\zeta} + \frac{\Phi \widetilde{B}(\eta)}{8\pi}\,, \\ &\text{with} \quad 0 < \zeta,\, \eta \le \xi\,, \end{aligned} \tag{91}$$

where we used the asymptotic properties (63), (87), and the (first) mean value theorem of integration with the intermediate values ζ and η, after a partial integration in the core region. We recognize two qualitatively different energy contributions; the first one from a "global" vortex, diverging logarithmically with the upper cutoff radius r_∞, which is characteristic for vortices in uncharged superfluids, and the second one from a "local" vortex, whose energy contribution has the standard "axis field" form $\Phi B(0)/8\pi$, which is typical for vortices in charged superfluids.

Using the decomposition into charged and uncharged superfluid subspaces via the charge projection tensors defined in (28), we can rewrite the first term in brackets in the form

$$\left[N^\Psi N_\Psi - \frac{(e_\Upsilon N^\Upsilon)^2}{(e^\Psi e_\Psi)}\right] = N_\perp^\Upsilon N_\Upsilon^\perp\,. \tag{92}$$

Concerning the second term in (91), if the magnetic field $\widetilde{B}(r)$ is slowly varying inside the vortex core, then we can approximately replace $\widetilde{B}(\eta) \approx \widetilde{B}(\xi)$, and use the explicit expression (66) with (67) and (65) to write

$$\widetilde{B}(\xi) = 4\pi\hbar\, (e_\Psi N_\parallel^\Psi) \left(1 - \frac{\Phi_{\rm core}}{\Phi}\right) \frac{K_0(x_0)}{x_0 K_1(x_0)}\,, \tag{93}$$

where $x_0 \equiv \xi/\ell$. In the extreme type–II limit, where the core structure becomes negligible, i.e. in the limit $\kappa = 1/x_0 \gg 1$, where $\Phi_{\rm core} \ll \Phi$, $K_0(\xi/\ell) \approx \ln(\ell/\xi)$, and $x_0 K_1(x_0) \approx 1$, equation (91) with (93) finally gives the simple expression for the vortex energy

$$U_{\rm vort} = \pi\hbar^2\, \mathbb{S}_{\Upsilon\Psi} \left[(N_\perp^\Upsilon N_\perp^\Psi) \ln\frac{r_\infty}{\xi} + (N_\parallel^\Upsilon N_\parallel^\Psi) \ln\frac{\ell}{\xi}\right]. \tag{94}$$

This "quasi separated" form clearly shows the respective contributions from a global vortex and a local vortex, but as mentioned above, even for vortices which have nonvanishing winding numbers only in either charged or uncharged constituents, there will generally be contributions from *both* terms, due to the entrainment matrix $\mathbb{S}$ involved in the projections.

VIII. DISCUSSION OF SOME APPLICATIONS

In order to illustrate the foregoing general results, we will in this section discuss some applications to well known standard examples of "realistic" superfluid systems, ordered by increasing complexity.

A. Single uncharged superfluid

Probably the simplest case are single, uncharged (isotropic) superfluids like ^{4}He. We note that vortices in ^{3}He show a much richer structure than in ^{4}He (e.g. see [12]), due to the anisotropic type of the microscopic Cooper pairing responsible for the superfluidity of ^{3}He. But the present approach should still be a good approximation at least for the ^{3}He–B superfluid [15], because sufficiently far from the vortex core the additional (anisotropic) degrees of freedom of the order parameter are "frozen" and the dynamics is again mainly governed by the phase φ^{Υ}.

a) at $\boldsymbol{T=0}$: In the case of a single superfluid at zero temperature, the "entrainment matrix" K_{XY} of (4) reduces to $K = n^0/\mu^0$, where n^α is the particle current and μ_α the momentum per particle of the superfluid. There are no normal fluids, so $\mathbb{S}$ of (24) is given trivially by $\mathbb{S} = K$. The charge vector vanishes, $e^{\Upsilon} = 0$, and the charge projection tensors are trivial, so $N_\perp^{\Upsilon} = N$ and $N_\parallel^{\Upsilon} = 0$. The vortex energy (91) then simply reduces to

$$U_{\text{vort}} = N^2 \pi \hbar^2 \frac{n^0}{\mu^0} \ln \frac{r_\infty}{\zeta} \,, \tag{95}$$

which is the same expression as found in [8] for the single superfluid. In the nonrelativistic limit, where $\mu^0 \to m$ and $n^0 \to n$ (where m is the rest mass of the superfluid particles, and n their number density), we recover the usual expression for the (hydrodynamic) superfluid vortex energy in the zero temperature limit (e.g. see [16]).

b) at $\boldsymbol{T\neq 0}$: In the case of a finite temperature, the system can be described as an effective superfluid–normal fluid mixture, where the normal fluid consists of the viscous gas of excitations in the superfluid. The superfluid and normal particle currents are $\boldsymbol{n}_{\text{S}}$ and $\boldsymbol{n}_{\text{N}}$, and their respective momenta per particle $\boldsymbol{\mu}^{\text{S}}$ and $\boldsymbol{\mu}^{\text{N}}$, say. There are no charged fluids, so $N_\parallel^{\text{S}} = 0$ and $N_\perp^{\text{S}} = N$. The entrainment matrix (5) reads

$$K_{\Upsilon\Psi} = \begin{pmatrix} K_{\text{SS}} & K_{\text{SN}} \\ K_{\text{NS}} & K_{\text{NN}} \end{pmatrix} , \tag{96}$$

and is decomposed in the mongrel representation of Sec. IV as $\mathbb{V} = 1/K_{\text{NN}}$, and $\mathbb{S} = K_{\text{SS}} - K_{\text{SN}}^2/K_{\text{NN}}$, so the vortex energy would simply be given by inserting this expression for $\mathbb{S}$ into equation (91). However, in order to compare this result to the usual expression for the vortex energy in superfluids at $T \neq 0$, we have to link the present entrainment formalism to the more common language of Landau's two–fluid model [13] that is expressed in terms of a "superfluid density" ρ_{S} and a "normal density" ρ_{N}. This "translation" has been achieved in a rigorous and extensive manner by Carter and Khalatnikov [17], but for the present purpose of an illustrative example, the following very simple argument should show in a sufficiently convincing way how to translate between the respective quantities. Namely, consider the total (spatial) momentum density T^{0i} (with $i = 1, 2, 3$) of the fluid mixture, for which from (16) we have $p^i \equiv T^{0i} = n_{\text{S}}^0 \mu^{\text{S}i} + n_{\text{N}}^0 \mu^{\text{N}i}$. Using the mongrel relations (22) and (23), this can be rewritten as $p^i = (\mu^{\text{S}0}\mathbb{S})\, \mu^{\text{S}i} + (n_{\text{N}}^0 \mathbb{V})\, n_{\text{N}}^i$. Now we introduce the normal velocity $v_{\text{N}}^i \equiv n_{\text{N}}^i / n_{\text{N}}^0$, which is the real mean velocity of the excitations, and the superfluid "pseudo–velocity" $\tilde{v}_{\text{S}}^i \equiv \mu^{\text{S}i}/\mu^{\text{S}0}$, which is not a "real" velocity in the sense of a particle transport. In the nonrelativistic limit, where $\mu^{\text{S}0}$ tends to the constant rest mass of the superfluid particles, $\mu^{\text{S}0} \to m_{\text{S}}$, the irrotationality property of superfluids (18) implies $\nabla^{[i} \tilde{v}_{\text{S}}^{j]} \approx 0$, in other words "rot $\vec{\tilde{v}}_{\text{S}} = 0$". In these variables the total momentum density now reads

$$p^i = \left[(\mu_0^{\text{S}})^2\, \mathbb{S}\right] \tilde{v}_{\text{S}}^i + \left[(n_{\text{N}}^0)^2\, \mathbb{V}\right] v_{\text{N}}^i \,. \tag{97}$$

Comparing this to the orthodox expression

$$p^i = \rho_{\rm S} \tilde{v}^i_{\rm S} + \rho_{\rm N} v^i_{\rm N} \,, \tag{98}$$

we can identify

$$\rho_{\rm S} = (\mu^{\rm S}_0)^2 \, \mathbb{S} \,, \quad \text{and} \quad \rho_{\rm N} = (n^0_{\rm N})^2 \, \mathbb{V} \,. \tag{99}$$

This is consistent with the additivity postulate $\rho = \rho_{\rm S} + \rho_{\rm N}$, namely using (25) we obtain the expression $\rho = n^0_{\rm S} \mu^{\rm S0} + n^0_{\rm N} \mu^{\rm N0}$, which effectively reduces to the total mass density in the Newtonian limit. In the present case we have $\mu^{\rm S0} \to m_{\rm S}$ for the superfluid, while $\mu^{\rm N0} \to 0$, as the normal fluid is identified with the gas of excitations, so the total mass density reduces to $\rho \to n_{\rm S} m_{\rm S}$.

In the nonrelativistic limit, expression (99) yields $\mathbb{S} = \rho_{\rm S}/m^2_{\rm S}$, and so the equation (91) for the vortex energy can explicitly be written as

$$U_{\rm vort} = \pi \hbar^2 N^2 \frac{\rho_{\rm S}}{m^2_{\rm S}} \ln \frac{r_\infty}{\zeta} \,, \tag{100}$$

in agreement with the well known result in Landau's two–fluid model (e.g. see [16]).

B. Two uncharged superfluids

In the next step, let us consider a vortex in a mixture of two uncharged superfluids, as first considered by Andreev and Bashkin [2] for a mixture of ^{3}He and ^{4}He. Again, at $T = 0$ there are no normal fluids, so we have

$$\mathbb{S}_{\Upsilon \Psi} = K_{\Upsilon \Psi} = \begin{pmatrix} K_{33} & K_{34} \\ K_{43} & K_{44} \end{pmatrix} . \tag{101}$$

The charge vector vanishes, $e^\Psi = 0$, and so $N^\Upsilon_\parallel = 0$ and $N^\Upsilon_\perp = \{ N^3, N^4 \}$. The expression (91) for the vortex energy in this case explicitly reads

$$\begin{aligned} U_{\rm vort} = \pi \hbar^2 \big[& (N^3)^2 K_{33} + (N^4)^2 K_{44} \\ & + 2 N^3 N^4 K_{34} \big] \ln \frac{r_\infty}{\zeta} \,. \end{aligned} \tag{102}$$

We see that there is a purely hydrodynamic interaction energy due to entrainment (i.e. not related to the condensation energy in the core) from the last term in brackets, which is either attractive or repulsive depending on the sign of the entrainment coefficient K_{34}.

C. Conventional Superconductors

When we consider cases with charged superfluids, the simplest example is already a two constituent system, because a second charged component is necessary to allow for global charge neutrality. This picture applies for example to conventional laboratory superconductors, where the charged superfluid (charge e^- and particle density n_-) consists of Cooper paired conduction electrons, while the second component is the "normal" background of positively charged ions (charge e^+ and particle density n_+). In the maximally symmetric and stationary situations considered in the present work, "normal" components are naturally restricted to uniform rotation (41), and therefore it makes no difference whether the normal component is actually a real "fluid" or a solid lattice like in the present example.

Because of the Cooper pairing mechanism, the fundamental superfluid charge carriers have to be considered as electron pairs, and therefore the charge per superfluid particle e^- should be twice the electron charge, i.e. $e^- = -2e$, and consequently the rest mass is $m^- = 2 m_{\rm e}$, where $m_{\rm e}$ is the electron rest mass. The entrainment matrix K_{XY}, defined in (5) can be written as

$$K_{XY} = \begin{pmatrix} K_{--} & K_{-+} \\ K_{+-} & K_{++} \end{pmatrix} , \tag{103}$$

and the transformation into the mongrel representation of Sec. IV yields $\mathbb{S} = K_{--} - (K_{-+})^2/K_{++}$ and $\mathbb{V} = 1/K_{++}$. The charge vector is just $e^\Upsilon = e^-$, and so $N^\Upsilon_\perp = 0$ and $N^\Upsilon_\parallel = N$.

The London field: In the simple case of a vortex–free state, i.e. with $N = 0$, there is nevertheless a nonvanishing uniform London field $B_\ominus$ if the superconductor is rotating (rotation rate Ω). Equation (59) for the London field immediately yields for this simple case $B_\ominus = -2\Omega(E/e^-)$, where E is the energy per superfluid particle, i.e. $E = -\mu_0^-$. If we choose a reference frame with $L \equiv \mu_z^- = 0$, i.e. comoving with the superconductor in z direction, then E can be identified with the (relativistic) chemical potential $\mu^- \equiv (-\mu_\alpha^- \mu_-^\alpha)^{1/2}$. In the Newtonian limit, where $\mu^- \approx m^-$, the conventional Newtonian chemical potential μ_{chem}^- is related to the relativistic chemical potential μ^- as

$$\mu^- = m^- \left(1 + \frac{\mu_{\text{chem}}^-}{m^-} + \mathcal{O}(\epsilon^2)\right), \tag{104}$$

where $\epsilon \equiv \mu_{\text{chem}}^-/m^- \ll 1$. The London field for a rotating superconductor can therefore be written in the form

$$B_\ominus = -2\Omega \frac{m^-}{e^-}\left(1 + \frac{\mu_{\text{chem}}^-}{m^-} + \mathcal{O}(\epsilon^2)\right). \tag{105}$$

It is well known that that the "entrainment" formalism for interacting constituents can equivalently be expressed in the more conventional (albeit sometimes less convenient) language of "effective masses" [2]. We see that in the case of two–constituent superconductors, the effect of entrainment (i.e. effective masses) cancels out in the expression (105) for the London field, which therefore depends quite naturally on the "bare" electron rest mass to charge ratio m^-/e^-, including a "relativistic" correction due to the finite chemical potential μ_{chem}^- of the electrons. We note that this cancellation only occurs for systems with a single superfluid constituent, where $\mathbb{S}$ is consequently a scalar and cancels out in (59). As soon as there is a second (interacting) superfluid constituent involved, as in the following example of neutron star matter, the London field *does* depend on the effective masses of the constituents. We further note that the present covariant treatment is intrinsically frame–independent, and contrary to the analysis of [18], we find that $B_\ominus$ does *not* depend on the chemical potential μ^+ of the "normal" component of positively charged ions.

A very crude estimate of the relativistic correction term μ_{chem}^-/m^- for a Nb superconductor at $T = 0$, taking μ_{chem}^- simply to be the Fermi energy of a free electron gas, yields a (positive) correction of the order 10^{-4}. This is in qualitative and nearly quantitative agreement with precision measurements performed on a rotating Nb superconductor [19]. But in order to effectively compare expression (105) with experimental results, a more careful estimation of μ_{chem}^- would be necessary.

Vortices: Now let us consider a vortex configuration, i.e. with $N \neq 0$. We see that a similar cancellation of the entrainment effect as for the London field (105) arises for the total flux of the vortex, which is seen by (65) to give the usual

$$\Phi = N\Phi_0, \quad \text{with} \quad \Phi_0 \equiv \frac{2\pi\hbar}{e^-}, \tag{106}$$

while the London penetration depth (64) *is* modified by entrainment, namely $l^{-2} = 4\pi(e^-)^2\mathbb{S}$. To write this more explicitly, we note that $\mathbb{S}$ can be written in the absence of entrainment as $\mathbb{S}^{(0)} = n_-/\mu^-$ and further $\mathbb{S}^{(0)} = (n_-/m^-)(1-\delta_{\text{rel}})$, where $\delta_{\text{rel}} \equiv \mu_{\text{chem}}^-/m^-$ is the same relativistic correction factor encountered in the expression for the London field (105). A nonvanishing entrainment interaction between the constituents will add an additional correction term δ_{entr} proportional to the matrix element K_{+-}, so that $\mathbb{S}$ can be written as $\mathbb{S} = (n_-/m^-)(1 + \delta_{\text{entr}} - \delta_{\text{rel}})$, and so the London penetration depth reads

$$\ell^{-2} = 4\pi(e^-)^2 \frac{n_-}{m^-}(1 + \delta_{\text{entr}} - \delta_{\text{rel}}). \tag{107}$$

The vortex energy is given by the "magnetic" term in (91) alone, due to $N_\parallel = N$ and $N_\perp = 0$, so we recover the usual "axis–field" expression

$$U_{\text{vort}} \approx \frac{\Phi \widetilde{B}(0)}{8\pi}, \tag{108}$$

which is seen in the more explicit form (94) (for the type–II limit, for simplicity) to depend on the effect of entrainment, namely

$$U_{\text{vort}} = N^2 \pi\hbar^2 \frac{n_-}{m^-}(1 + \delta_{\text{entr}} - \delta_{\text{rel}}) \ln\frac{\ell}{\xi}, \tag{109}$$

but as the total vortex energy $U = U_{\text{vort}} + U_{\text{con}}$ also depends on the largely unknown condensation energy of the core, the relativistic and entrainment corrections in this expression seem unlikely to be of observable interest.

D. Neutron star matter (Outer core)

In this last example we consider the case of a (cold) degenerate plasma consisting of neutrons, protons and electrons in β equilibrium, as relevant for the outer core of neutron stars (i.e. at densities $\gtrsim$ nuclear density). In this case one usually assumes that there is an important entrainment between neutrons and protons due to their strong interactions, while the entrainment with electrons is generally supposed to be negligible. We will follow this assumption and denote the entrainment matrix as

$$K_{XY} = \begin{pmatrix} K_{\rm nn} & K_{\rm np} & 0 \\ K_{\rm pn} & K_{\rm pp} & 0 \\ 0 & 0 & K_{\rm ee} \end{pmatrix} . \tag{110}$$

The calculations of the superfluid gaps for this neutron star matter generally suggest (see for example [20]) that the protons will be superconducting and the neutrons superfluid, while the electrons remain "normal", so this system would represent a superconducting–superfluid–normal mixture. The matrices of the mongrel representation of Sec. IV for this system read $\mathbb{M}=0$, $\mathbb{V}=1/K_{\rm ee}$ and

$$\mathbb{S}_{\Upsilon\Psi} = \begin{pmatrix} K_{\rm nn} & K_{\rm np} \\ K_{\rm pn} & K_{\rm pp} \end{pmatrix} , \tag{111}$$

and we define an "entrainment coefficient" $\alpha \equiv K_{\rm np}/K_{\rm pp}$. For this system the charge vectors and projections are nontrivial, namely

$$e^{\Psi} = \begin{pmatrix} 0 \\ q \end{pmatrix} , \quad e_{\Psi} = q K_{\rm pp}\,(\alpha,\,1)\ , \tag{112}$$

$$\eta^{\Upsilon}_{\Psi} = \begin{pmatrix} 0 & 0 \\ \alpha & 1 \end{pmatrix} , \quad \gamma^{\Upsilon}_{\Psi} = \begin{pmatrix} 1 & 0 \\ -\alpha & 0 \end{pmatrix} , \tag{113}$$

where q is the charge of a proton Cooper pair, i.e. $q=2|e|$, and we further have

$$N^{\Upsilon}_{\parallel} = (N^{\rm p} + \alpha N^{\rm n}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \tag{114}$$

$$N^{\Upsilon}_{\perp} = N^{\rm n} \begin{pmatrix} 1 \\ -\alpha \end{pmatrix} . \tag{115}$$

The London penetration depth (64) is

$$\ell^{-2} = 4\pi q^2 K_{\rm pp}\,, \tag{116}$$

and the vortex flux (65) is found as

$$\Phi = (N^{\rm p} + \alpha N^{\rm n})\Phi_0\,, \quad \text{with} \quad \Phi_0 = \frac{2\pi\hbar}{q}\,, \tag{117}$$

in agreement with earlier results in the literature [5,3,6]. The vortex energy in the type–II limit (94) reads

$$\begin{aligned} U_{\rm vort} = {}& \pi\hbar^2 (N^{\rm n})^2 \left[(K_{\rm nn} - \alpha^2 K_{\rm pp}) \ln\frac{r_\infty}{\xi} + \alpha^2 K_{\rm pp} \ln\frac{\ell}{\xi} \right] \\ & + \pi\hbar^2 (N^{\rm p})^2 K_{\rm pp} \ln\frac{\ell}{\xi} \\ & + 2\pi\hbar^2 N^{\rm n} N^{\rm p} K_{\rm np} \ln\frac{\ell}{\xi}\,. \end{aligned} \tag{118}$$

Similar to the case of a mixture of two uncharged superfluids, we see that the total vortex energy consists of a pure n–vortex term, a pure p–vortex term (each of which is modified by the entrainment), while the last term represents an attractive or repulsive (depending on the sign of $K_{\rm np}$) interaction term with respect to infinite separation. It has been suggested [21] that the effect of entrainment between neutrons and protons could energetically favor a "vortex cluster" structure (i.e. a neutron vortex surrounded by a dense lattice of proton vortices) with respect to a single neutron vortex. This question can strictly speaking not be addressed in the present framework of perfectly axially symmetric configurations, and will be subject of future investigation, but the energy of a single n–vortex ($N^{\mathrm{p}} = 0$, $N^{\mathrm{n}} \neq 0$) is seen from expression (118) to be of the same order of magnitude if not smaller than in the absence of entrainment ($\alpha \to 0$), i.e. $U_{\mathrm{vort}}^{(0)} = \pi\hbar^2 (N^{\mathrm{n}})^2 K_{\mathrm{nn}}^{(0)} \ln(r_\infty/\xi)$. Any configuration containing more vortices is therefore rather expected to have a higher energy, but the possibly attractive interaction term in (118) could lead to an effective "clustering" of already present vortices, namely a n–vortex that "accretes" p–vortices until saturation.

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[1] I.M. Khalatnikov, Sov. Phys. JETP **5** (4), 542-545 (1957).
[2] A.F. Andreev, E.P. Bashkin, Sov. Phys. JETP **42 (1)**, 164 (1975).
[3] G.A. Vardanian, D.M. Sedrakian, Sov. Phys. JETP **54** (5), 919-921 (1981).
[4] G. Mendell, L. Lindblom, Ann. Phys. **205**, 110 (1991).
[5] D.M. Sedrakian, K.M. Shahabasian, Astrophysics **16**, 417-422 (1980).
[6] M.A. Alpar, S.A. Langer, J.A. Sauls, Astrophys. J. **282**, 533 (1984).
[7] G. Mendell, Astrophys. J. **380**, 515 (1991).
[8] B. Carter, D. Langlois, Phys. Rev. **D52**, 4640 (1995).
[9] B. Carter, R. Prix, D. Langlois, (`cond-mat/9910240`).
[10] B. Carter, in *Relativistic Fluid Dynamics (Noto, 1987)*, Lecture Notes in Mathematics **1385**, ed. A. Anile and M. Choquet–Bruhat (Springer–Verlag, Heidelberg 1989), pp. 1–64.
[11] B. Carter, D. Langlois, Nucl. Phys. **B 531**, 478 (1998).
[12] V.B. Eltsov, M. Krusius, in *Topological Defects and the Non–Equilibrium Dynamics of Symmetry Breaking Phase Transitions*, proceedings of the NATO Advanced Study Institute, Les Houches, 1999, edited by Y.M. Bunkov and H. Godfrin, (Kluwer Academic Publisher, Dortrecht, 2000), pp. 325–344.
[13] L.D. Landau and E.M. Lifshitz, *Course of Theoretical Physics*, Vol. 6, *Fluid Mechanics*, (Pergamon, Oxford, 1959).
[14] B. Carter, D. Langlois, Phys. Rev. **D51**, 5855 (1995).
[15] G.E. Volovik, private communication.
[16] D.R. Tilley and J. Tilley, *Superfluidity and Superconductivity*, (I.O.P, Bristol, 1990).
[17] B. Carter, I.M. Khalatnikov, Rev. Math. Phys. **6** (2), 277-304 (1994)
[18] M. Liu, Phys. Rev. Lett. **81**, 3223 (1998).
[19] J. Tate, B. Cabrera, S.B. Felch, J.T. Anderson, Phys. Rev. Lett. **62**, 845 (1989).
[20] M. Baldo, J. Cugnon, A. Lejeune, U. Lombardo, Nucl. Phys. **A 536**, 349 (1992).
[21] A.D. Sedrakian, D.M. Sedrakian, Astrophys. J. **447**, 305-323 (1995).